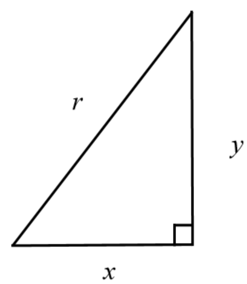
Exercise 24

A particle moves along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $\left(\frac{1}{3}, 1\right)$, its x-coordinate increases at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the particle to the origin changing at this instant?

Solution

At any time, the distance from the origin to the particle is given by the Pythagorean theorem.



$$r^2 = x^2 + y^2$$

Take the derivative of both sides with respect to time by using the chain rule.

$$\begin{aligned}
\frac{d}{dt}(r^2) &= \frac{d}{dt}(x^2 + y^2) \\
2r \cdot \frac{dr}{dt} &= 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} \\
r\frac{dr}{dt} &= x\frac{dx}{dt} + y\frac{dy}{dt} \\
&= x\frac{dx}{dt} + \left(2\sin\frac{\pi x}{2}\right)\frac{d}{dt}\left(2\sin\frac{\pi x}{2}\right) \\
&= x\frac{dx}{dt} + \left(2\sin\frac{\pi x}{2}\right)\left(2\cos\frac{\pi x}{2}\right)\cdot\frac{d}{dt}\left(\frac{\pi x}{2}\right) \\
&= x\frac{dx}{dt} + 2\left(2\sin\frac{\pi x}{2}\cos\frac{\pi x}{2}\right)\cdot\left(\frac{\pi}{2}\cdot\frac{dx}{dt}\right) \\
&= x\frac{dx}{dt} + 2(\sin\pi x)\left(\frac{\pi}{2}\cdot\frac{dx}{dt}\right) \\
&= x\frac{dx}{dt} + \pi\sin\pi x\frac{dx}{dt} \\
&= (x + \pi\sin\pi x)\frac{dx}{dt}
\end{aligned}$$

Since the x-coordinate is increasing at a rate of $\sqrt{10}$ cm/s, $dx/dt = \sqrt{10}$.

$$r\frac{dr}{dt} = (x + \pi \sin \pi x)\sqrt{10}$$

Solve for dr/dt.

$$\begin{aligned}\frac{dr}{dt} &= \frac{\sqrt{10}\,(x + \pi \sin \pi x)}{r} \\ &= \frac{\sqrt{10}\,(x + \pi \sin \pi x)}{\sqrt{x^2 + y^2}}\end{aligned}$$

Therefore, as the particle passes through the point $\left(\frac{1}{3}, 1\right)$, the rate that the distance from the particle to the origin is changing is

$$\left.\frac{dr}{dt}\right|_{\substack{x=1/3 \\ y=1}} = \frac{\sqrt{10}\left(\frac{1}{3} + \pi \sin \frac{\pi}{3}\right)}{\sqrt{\left(\frac{1}{3}\right)^2 + (1)^2}} = \frac{3\pi\sqrt{3}}{2} + 1 \approx 9.1621\ \frac{\text{cm}}{\text{s}}.$$